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DISC-SHAPED EQUILIBRIUM CRACKS†

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Novozhilov's approach [1,2] for investigating equilibrium cracks in the plane problem is extended to a disc-shaped crack. A crack, circular in plan, which is acted upon either by two normal and oppositely directed point forces, applied at the centre of the surfaces, or a uniform uniaxial tension, is considered. A narrow zone of cohesion at the crack edge is taken into account. The possible range of diameters of equilibrium cracks is established in each case. © 2000 Elsevier Science Ltd. All rights reserved.

In developing the Griffith–Irwin theory of cracks, Novozhilov [1, 2] established the possible range of lengths of equilibrium cracks. This investigation correlates with the well-known Thomson theory on lattice trapping and the range of external forces which ensure the equilibrium state of cracks [3, 4]. The Thomson–Novozhilov approach enabled a solution of a number of applied problems to be obtained [5, 6], which stimulates further investigation in this area.

Consider an unbounded elastic isotropic body with a disc-shaped crack, circular in plan, $\Gamma = \{(x_1, x_2, x_3) : x_3 = 0, x_1^2 + x_2^2 \leq a^2\}$ (Fig. 1). We will assume that, in general, forces $\sigma_{33} = p(x_1, x_2)$ are applied to the surface Γ, symmetrical about the x_3 axis, constant cohesion forces σ_0 act at the crack edge in a narrow circular region $\Gamma_0 = \{(x_1, x_2, x_3) : x_3 = 0, b^2 \leq x_1^2 + x_2^2 \leq a^2\}$, and a force $\sigma_{33}^\infty = \sigma$ acts at infinity.

In the cohesion region Γ_0 the displacement of points of the crack surface w in the direction of the x_3 axis satisfies the condition.

$$w(\rho) \leq w_0, \quad b \leq \rho \leq a, \quad \rho = \sqrt{x_1^2 + x_2^2} \tag{1}$$

Following Novozhilov [1, 2], we will use the condition for the crack to propagate in the form.

$$\int_a^{a+D} \sigma_{33}(\rho) d\rho \geq D\sigma_f \tag{2}$$

In (1) and (2) w_0, D, σ_f are constants of the material, $2w_0$ is the limit separation of points of the crack surfaces, between which the cohesion forces act, D is the size of the fracture zone at the crack edge, and σ_f is the breaking stress for uniaxial tension. In brittle fracture, the parameter D is given approximately by the equation [7]

$$D = 2K_{1c}^2 / (\pi\sigma_f^2) \tag{3}$$

where K_{1c} is the fracture viscosity.

Cracks of diameter d_g and d_c correspond to the equality sign in the first relation in (1) and in (2) respectively. Depending on the load, the range of equilibrium cracks is determined by one of the following double inequalities

$$d_c \leq d \leq d_g, \quad d_g \leq d \leq d_c \tag{4}$$

We will obtain expressions for d_g and d_c in two model problems, assuming,with the notation $\Delta = a - b$, that

$$\Delta/a \ll 1, \quad D/a \ll 1 \tag{5}$$

The crack is opened by a pair of normal and oppositely directed point forces P, applied at the centre of the surfaces Γ and there are no stresses at infinity (Fig. 2). In this case $p(x_1, x_2) = -P\delta(\rho)$ and $\sigma = 0$.

Using the solution constructed previously [8, 9] to investigate the behaviour of a disc-shaped crack in the Leonov–Panasyuk–Dugdale model, and also Green's function [10, 11] and the superposition method, we arrive at the following relations for w and σ_{33} when $x_3 = 0$

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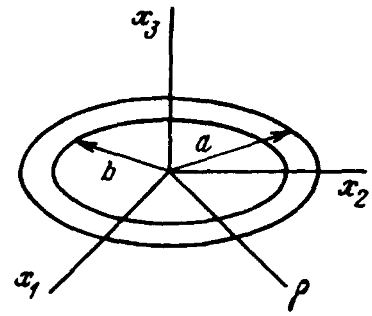

Fig. 1.

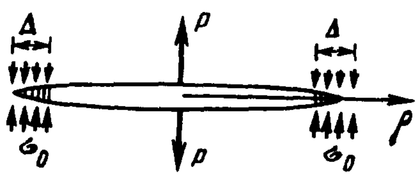

Fig. 2.

$$w(\rho)=\frac{2PH}{\pi\rho}\operatorname{arctg}\frac{\sqrt{a^2-\rho^2}}{\rho}-F(\rho,c) \tag{6}$$

$$H=\frac{1-\nu^2}{\pi E},\quad F(\rho,c)=4H\sigma_0\int_c^a\sqrt{\frac{t^2-b^2}{t^2-\rho^2}}dt,\quad c=\begin{cases}b, & 0\leq\rho\leq b\\ \rho, & b\leq\rho\leq a\end{cases}$$

$$\sigma_{33}(\rho)=\frac{P_a}{\pi^2\rho^2\sqrt{\rho^2-a^2}}-\frac{2\sigma_0}{\pi}\left[\frac{\sqrt{a^2-b^2}}{\sqrt{\rho^2-a^2}}-\arcsin\sqrt{\frac{a^2-b^2}{\rho^2-a^2}}\right],\quad \rho>a \tag{7}$$

(υ is Poisson's ratio and E is Young's modulus).

When $\rho = b$, from (6), taking the first condition in (5) into account, we obtain the equation

$$\frac{4P}{\pi\delta_0 d^2}\left(\frac{d}{D}\right)^{1/2}=\frac{\beta}{2\sqrt{\alpha}}+2\sqrt{\alpha},\quad \alpha=\frac{\Delta}{D},\quad \beta=\frac{w_0}{HD\sigma_0} \tag{8}$$

or

$$2\left(\frac{d_g}{d}\right)^{3/2}=U+\frac{1}{U},\quad U^2=\frac{\beta}{4\alpha},\quad d_g=\left(\frac{4HP^2}{\pi^2 w_0\sigma_0}\right)^{1/3} \tag{8}$$

It obviously follows from the last equation that $d \leq d_g$.

Substituting (7) into (2), taking conditions (5) into account we arrive at the following condition for the crack to propagate

$$\frac{4P}{\pi\sigma_0 d^2}\left(\frac{d}{D}\right)^{1/2}-2\sqrt{\alpha}+2A(\alpha)\geq\frac{\pi\sigma_f}{\sigma_0}$$
$$A(\alpha)=\arcsin\sqrt{\frac{\alpha}{1+\alpha}}-\alpha\arcsin\sqrt{\frac{1}{1+\alpha}} \tag{10}$$

Assuming $d = d_c$ in (8) and (10), we obtain an equation for finding the width of the cohesion zone Δ, corresponding to a critical crack of diameter d_c,

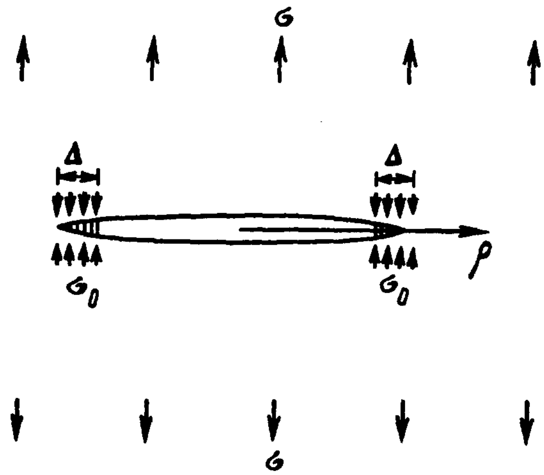

Fig. 3.

$$\frac{\beta}{2} = \frac{\pi\sigma_f}{\sigma_0}\sqrt{\alpha} - 2\sqrt{\alpha}A(\alpha) \tag{11}$$

After determining α from (11), the critical diameter d_c of the crack can be found from (8) for a fixed value of the force P. The range of equilibrium cracks is determined by the first double inequality in (4).

The crack is opened by forces σ at infinity. The surface of the crack, with the exception of the cohesion zone, is free, i.e. $p(x_1, x_2) \equiv 0$ (Fig. 3).

In this case the solution of the problem in the $x_3 = 0$ plane has the form [8, 9]

$$w(\rho) = 4H\sigma\sqrt{a^2 - \rho^2} - F(\rho, c) \tag{12}$$

$$\sigma_{33}(\rho) = \frac{2\sigma_0}{\pi}\left[\frac{as - \sqrt{a^2 - b^2}}{\sqrt{\rho^2 - a^2}} + \frac{\pi s}{2} - s\arcsin\frac{a}{\rho} + \arcsin\sqrt{\frac{a^2 - b^2}{\rho^2 - b^2}}\right] \tag{13}$$

$$\rho > a, \quad s = \sigma/\sigma_0$$

As in the previous case, from (12) when $\rho = b$ we obtain a relation similar to (8),

$$\frac{2\sigma}{\sigma_0}\left(\frac{d}{D}\right)^{1/2} = \frac{\beta}{2\sqrt{\alpha}} + 2\sqrt{\alpha} \tag{14}$$

or

$$2\left(\frac{d}{d_g}\right)^{1/2} = U + \frac{1}{U}, \quad d_g = \frac{w_0\sigma_0}{H\sigma^2} \tag{15}$$

In this case it follows from (15) that $d \geqslant d_g$. Substituting (13) into criterion (2) and taking relation (5) into account we obtain a condition similar to (10),

$$\frac{2\sigma}{\sigma_0}\left(\frac{d}{D}\right)^{1/2} - 2\sqrt{\alpha} + 2A(\alpha) \geqslant \frac{\pi\sigma_f}{\sigma_0} \tag{16}$$

Hence from (14) with $d = d_c$ we arrive at the same equation (11) for finding Δ. The range of equilibrium cracks is determined in this problem by the second double inequality in (4).

Note that in the same way as previously [2] we can introduce a parameter $\varkappa = d_-/d_+$ which is independent of the external load and is a certain parameter of the material (see also [4]). Here d_+ and d_- are the upper and lower boundaries of the values of the diameter of the equilibrium cracks respectively.

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